T0. Semi-graphs of Anabelioids

· Anabelioids:

anabelioid $\overset{\mathrm{dfn}}{=}$ multi-Galois category
↳ allow for fin. many connected components

conn'd anab. $\overset{\mathrm{dfn}}{=}$ Galois category

point of new terminology: we want to think of anab'ds as geometric objects.

... e.g. we want to consider fin. ét coverings of anab'ds.

(conn'd) anab'd $\mathcal{B}(\Pi)$: finite sets with cont. Π-action
Π: prof. gp.
$\wr\wr$
"Π up to inner automs" ⟵ bp indeterminacy

$H \subseteq \Pi$ open s/gp.
$\rightsquigarrow \mathcal{B}(H) \xrightarrow{\text{fin. ét.}} \mathcal{B}(\Pi)$
↳ i.e. "$(\ )^{*}$" in opp. direction

note: a priori is 1-morphism in a 2-cat.
$\rightsquigarrow$ we want to work with mors. in a 1-cat.
$\Downarrow$
possible if Π is slim
(i.e., $Z_{\Pi}(H) = \{1\}$ $\forall_{\text{open}} H \subseteq \Pi$)
$\Downarrow$
$\mathcal{B}(H) \to \mathcal{B}(\Pi)$
$\mathrm{aut}(\ ) = \{1\}$

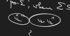
· Semi-graphs of anabelioids

closed / open

e.g. dual graph of stable curve / alg. cl. fld. (pt'd)
vertices ⟷ irred. comps.
closed edges ⟷ nodes
open edges ⟷ cusps

semi-gr. of anab'ds. $\mathcal{G}$:
· underlying semi-graph Γ
· $\forall v \in \mathrm{Vert}(\Gamma)$, conn. anab'd $\mathcal{G}_v$
· $\forall e \in \mathrm{Edge}(\Gamma)$, conn. anab'd $\mathcal{G}_e$
· $\forall$ b of an $e \in \mathrm{Edge}(\Gamma)$, abutting to $v \in \mathrm{Vert}(\Gamma)$, $\mathcal{G}_e \to \mathcal{G}_v$

semi-grs. of anab'd of PSC-type
↳ pt'd stable curve / k
pro-Σ, where $\Sigma \subseteq$ Primes s.t. $\mathrm{char}(k) \notin \Sigma$



$\mathcal{B}(\hat{\mathbb{Z}}^{\Sigma})$ not slim
$\mathcal{G}_e \to \mathcal{G}_v$
$\mathcal{B}(\hat{\mathbb{Z}}^{\Sigma}) \hookrightarrow \mathcal{B}(\hat{\Pi}^{\Sigma}_{(g_v, r_v)})$
slim
↳ suff. to work with mors. in a 1-cat.

fin. ét. covering of a semi-gr. of anab'ds.
fin. ét covering of each $\mathcal{G}_v$, $\mathcal{G}_e$ together with gluing isoms.

· Zariski's main thm. of semi-graphs

immersion:

injection on branches

embedding: isom onto a s/semi-graph

free gp. of finite rk. $F = \pi_1^{\mathrm{top}}(\Gamma)$ fin. gp.
$\cup$
fin. gen. s/gp. $H = \pi_1^{\mathrm{top}}(\Gamma^{*}) = \pi_1^{\mathrm{top}}(\Gamma^{\dagger})$
for some $\Gamma^{*} \to \Gamma$ covering
$\cup$
fin. s/gp. $\Gamma^{\dagger}$
immersion

$\Gamma^{\dagger} \longrightarrow \Gamma$
emb. $\Gamma^{\ddagger}$ f. ét.

F
$\cup$ fin. index
$H \subseteq G$
$\langle g_1, \dots, g_j \rangle$ $\langle g_1, \dots, g_n \rangle$

· fin. gps. acting on trees.
(cf. Serre, Trees)
↳ semi-graph w/o loops.
$\Leftrightarrow \pi_1^{\mathrm{top}} = \{1\} \Leftrightarrow H_1 = \{1\}$

... cf. [CbTpIV], §1,
$\exists$ a profinite analogue

· $\mathcal{G}$ semi-gr. of anab'ds. of PSC-type
f. ét. coverings
$\mathcal{B}(\mathcal{G}) = \mathcal{B}(\Pi_{\mathcal{G}})$ ⟵ well def'd up to inner autom
$\mathcal{G} \overset{\text{tp'd covering}}{\rightsquigarrow} \mathcal{B}^{\mathrm{tp}}(\mathcal{G}) = \mathcal{B}^{\mathrm{tp}}(\Pi^{\mathrm{tp}}_{\mathcal{G}})$

tp'd covering: subcovering of
(f. ét covering) ∘ (top covering)
↳ of underlying semi-graph

$\forall v \in \mathrm{Vert}(\mathcal{G}) \rightsquigarrow \Pi_v \subseteq \Pi^{\mathrm{tp}}_{\mathcal{G}} \subseteq \hat{\Pi}^{\mathrm{tp}}_{\mathcal{G}} \simeq \Pi_{\mathcal{G}}$
$\forall e \in \mathrm{Edge}(\mathcal{G}) \rightsquigarrow \Pi_e \subseteq \dots\dots \Pi_{\mathcal{G}}$
well def'd up to conjugacy
↳ verticial s/gp.
↳ edge-like s/gps

$v \neq w \rightsquigarrow$ non-conj. Π_v, Π_w
$e \neq f \rightsquigarrow$ non-conj. Π_e, Π_f
$C_{\Pi_{\mathcal{G}}}(\Pi_v) = \Pi_v$
$C_{\Pi_{\mathcal{G}}}(\Pi_e) = \Pi_e$

↳ obtained by fixing together coverings

T1. Motivation from differential topology and arithmetic

- Semi-graphs of anab'ds and config. spaces

X hyp. curve / alg. cl. fld. k

$X_n = \underbrace{X \times \dots \times X}_{n} \setminus \{\text{diagonals } \text{"} x_i = x_j \text{"} ,\ i \neq j,\ i,j \in \{1, \dots, n\}\}$

n^{th} config. sp.

$\rightsquigarrow \Pi_X = \Pi_{X_1}$, $\Pi_{X_n} \twoheadrightarrow \Pi_m$

$X_n \twoheadrightarrow X_m \quad n \geq m$

pointed stable log curve $X^{\log}/k$ of type (g, r)

$$\begin{array}{ccc} X_n^{\log} & \longrightarrow & \overline{\mathcal{M}}^{\log}_{g, r+n} \\ \downarrow & \square & \downarrow \\ [X^{\log}] & \longrightarrow & \overline{\mathcal{M}}^{\log}_{g,r} \end{array}$$

n^{th} log. config. space

ordered

$X_n^{\log} \twoheadrightarrow X_m^{\log}$

$\underline{\underline{\Pi_n}} := \Pi_{X_n^{\log}} \twoheadrightarrow \Pi_{X_m^{\log}}$

$\underline{\underline{\Pi_n}} \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \dots \twoheadrightarrow \Pi_1$

$\pi_1^{\log}$($\exists$ ptl'd stable log curve)

fiber

$\rightsquigarrow$ Semi-gr. of anab'ds of PSC-type

$\rightsquigarrow$ tower of . . . . .

DNB Thm. (diff. top)

$\frac{\text{top}}{\text{ll}}$, $\mathcal{X}$

$\exists$ Aut arises from a $\underline{\underline{\text{self-homeom.}}}$ of

$\Downarrow$

$\mathcal{X}$?

$\mathcal{X}_n$?

$\Downarrow$

$\text{Out}(\Pi^{\text{top}}_{\mathcal{X}_n})$ "$\simeq$" $\text{Out}(\Pi^{\text{top}}_{\mathcal{X}})$

"cuspidalization problem"

$\underline{\text{Out}(\Pi_n)}$ "$\simeq$" $\text{Out}(\Pi_m)$

- Π_3: log. config. sp. "$\simeq$" tripod

$\wr\wr$

ring str.

even though one works with purely comb. / gp-th. obj (geom. fund. gps), one can still naturally reconstruct $\overline{\mathbb{Q}}$,

$G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$\subseteq \underline{GT}$

T2. Combinatorial anabelian results

$X^{\log}$ in universal situation,

$\prod_{\text{nodes}} \hat{\mathbb{Z}}^{\Sigma}(1) \rightarrow \text{Out}(\Pi_{X^{\log}})$

nodes

$\uparrow$

I

$\pi_1^{\log}(\ )$

log pt. $\longrightarrow \underline{\overline{\mathcal{M}}^{\log}_{g,r}}$

$\hat{\mathbb{Z}}^{\Sigma} \cong I \subseteq \prod_{\text{nodes}} \hat{\mathbb{Z}}^{\Sigma}(1)$

$\underline{\underline{\text{vector}}}$ $(\hat{\mathbb{Z}}^{\Sigma})^{\times}$

$(\ ,\ ,\ )$: IPSC

$(\ ,\ ,\ )$: NN

e.g. $\underline{\underline{(1, -1)}}$

T3. Combinatorial cuspidalization and "FC = F" results

- $\text{Out}^F(\Pi_n)$, $\text{Out}^C(\Pi_n)$

F: preserves fibers/gps., i.e., $\text{Ker}(\Pi_n \twoheadrightarrow \Pi_m)$

C: $\Pi_n \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \Pi_{n-2} \twoheadrightarrow \dots \twoheadrightarrow \Pi_1$

preserves these kernels

$\text{Ker} = \pi_1(\text{fiber})$

$\cup$

and $\longrightarrow$ { inertia s/gps. of cusps

$\cup$

cuspidal edge-like s/gps.

$FC = F \wedge C$

sometimes autom.

T4. Arithmetic applications of combinatorial cuspidalization over number fields and mixed-characteristic local fields

$\text{Out}^{G}(\Delta) \subseteq \text{Out}(\Delta)$

preserve dual graph str. of special fibers of fct. coverings of X

(cf T0) $\|$

$\text{Out}(\Delta^{\text{tp}})$

$\text{Out}^{M}(\Delta) \subseteq \text{Out}^{G}(\Delta)$

preserve $\underline{\text{metrized}}$ dual graph str. of - - (- - - .

"node $\mapsto$ index of node"

DVR

$\pi \in A[[x, y]]/(xy - \pi^a)$

uniformizer

a pos. int.

$\cup$

index

T5. Synchronization of cyclotomes, profinite Dehn twists, and geometric monodromy anabelian results

· Synchronization of cyclotomes ($\twoheadrightarrow$ Quot$(\cdot)$) $\cong \hat{\mathbb{Z}}(1)$

$\mathcal{G}$

semi-gr. of anab'ds

$\chi_{cyc}: \mathrm{Aut}(\mathcal{G}) \to \hat{\mathbb{Z}}^\times$
$\cup$
open sgp.

e.g. · edge-like s/gps. $\Pi_e \subseteq \Pi_{\mathcal{G}}$ (cuspidal, nodal)

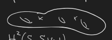
· $H^2_c(\mathcal{G}, \hat{\mathbb{Z}})^\vee$ ← compact support at cusps

· $H^2_c(\mathcal{G}_v, \hat{\mathbb{Z}})^\vee$ ← compact support at cusps, nodes that abut to v

$\exists$ synchronization isoms. btw. … that are intrinsically assoc'd to $\mathcal{G}$

"profinite versions of local modules of orientations of a top. surf."

S



x
$z \mapsto H^2(S, S\setminus\{z\})$

· profinite Dehn twists (prof. analogue of Dehn twists)

conj. by α $\circlearrowleft \Pi_v$ id $\circlearrowleft \Pi_w$

$\mathcal{G}$ semi-gr. of anab'ds.

$\mathrm{Dehn}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$

s.t. "id" on underlying semi-graph, "id" on each $\mathcal{G}_v$

$\cong \prod_{e \in \mathrm{Node}(\mathcal{G})} \Pi_e \geq I \cong \hat{\mathbb{Z}}^\Sigma$

$\Pi_e \cong \hat{\mathbb{Z}}^\Sigma$ IPSC NN

$\alpha \in \Pi_e$

$\mathcal{C}_{g,r}$
$\downarrow$
$\mathcal{M}_{g,r} \rightsquigarrow \Pi_{\mathcal{M}_{g,r}} \overset{\text{out}}{\curvearrowright} \underline{\Pi_{g,r}} =: \Pi$

$\rightsquigarrow \Pi_{\mathcal{M}_{g,r}} \to \mathrm{Out}(\Pi)$

T6. Tripod synchronization and the tripod homomorphism

log config. spaces $X^{\log}_n$ assoc'd to $X^{\log}$

blow-ups $\supseteq$ $\underline{\underline{\text{tripod}}}$

new tripod

$X^{\log}_3 \to X^{\log}_2 \supseteq$ diag. divisor $\to X^{\log}_1$
$\cup$
central tripod

$X^{\log}$
$\cup$

T7. Structure theory and arithmetic subgroups of the Grothendieck-Teichmuller group

$\Pi_n \twoheadrightarrow \Pi_m \twoheadrightarrow \Pi_1$

$(g,r) = \underline{(0,3)}, \underline{(1,1)}$ } in these cases, $\exists$ also generalized proj. mors

↳ if exclude these cases, $\exists_n$ proj. to Π_{n-1}
$(0,3) \rightsquigarrow \exists_{n+3}$ …
$(1,1) \rightsquigarrow \exists_{n+1}$ …

"Π_U" $= H \twoheadrightarrow \underline{\underline{\Pi^{\mathrm{trp}}}}$
$\cap$
$\underline{\underline{\Pi^{\mathrm{trp}}}}$ $\curvearrowleft$ $\underline{\text{Sch-th.}}$ $\curvearrowleft \mathfrak{S}_{n+3}$

$U \overset{\text{open}}{\subseteq} X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$

COF: cofiltered

$\exists$ $\to \cdot \to \cdot$, $\to \cdot \to$

RGC: "rel. GC"
$\exists !$

$\rightsquigarrow \overline{\mathbb{Q}}_{BGT}$: copy of $\overline{\mathbb{Q}}$